CS 58500 Theoretical Computer Science Toolkit (Spring 2026) — Problem Set # 1

{{Your name}} — **Due: Feb 12th**

Problem 1: Log-sum Inequality. (20=15+5 points)

1. Let $\{a_1, \dots, a_N\}$ and $\{b_1, \dots, b_N\}$ be two sets of positive real numbers. Use Jensen's inequality to prove the following inequality.

$$\sum_{i=1}^{N} a_i \ln \frac{a_i}{b_i} \geq A \ln \frac{A}{B},$$

   where $A := \sum_{i=1}^{N} a_i$ and $B := \sum_{i=1}^{N} b_i$. Furthermore, equality holds if and only if a_i/b_i is identical for all $i \in \{1, \dots, N\}$.

2. Let $\mathcal{X}$ be a finite set and $P : \mathcal{X} \to [0,1]$ and $Q : \mathcal{X} \to [0,1]$ be two probability distributions on $\mathcal{X}$ such that for any $x \in \mathcal{X}$, $Q(x) \neq 0$. The relative entropy from Q to P is defined as follows:

$$D(P \| Q) := \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.$$

   Show that for any P and Q, it holds that $D(P\|Q) \geq 0$. Moreover, state when $D(P\|Q) = 0$.

Solution.

Problem 2: Chernoff–Hoeffding Inequality. (10 pts)

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with $\mathbb{E}[X_i] = p$. Let $Z := \sum_{i=1}^n X_i$. Prove the following Chernoff-Hoeffding inequalities:

$$\Pr\big[Z \geq (p+\varepsilon)n\big] \leq \exp\big(-n\, D(p+\varepsilon\|p)\big) \qquad \forall\, \varepsilon \in (0, 1-p),$$
$$\Pr\big[Z \leq (p-\varepsilon)n\big] \leq \exp\big(-n\, D(p-\varepsilon\|p)\big) \qquad \forall\, \varepsilon \in (0, p),$$

where the binary relative entropy is

$$D(p\|q) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}.$$

Solution.

Problem 3: Tight Estimation: Central Binomial Coefficient. (30 pts)

We will learn a new powerful technique to prove tight inequalities. As a representative example, we will estimate the central binomial coefficient. For positive integer n, we will prove that

$$L_n \le \binom{2n}{n} \le U_n,$$

where

$$L_n := \frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} \qquad U_n := \frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{46n}\right)}}.$$

To prove these bounds, we will use the following general strategy.

1. Define the following two sequences

$$\left\{a_n := \binom{2n}{n}/U_n\right\}_n \qquad \left\{b_n := \binom{2n}{n}/L_n\right\}_n$$

2. *Prove the following limit.*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\binom{2n}{n}}{4^n/\sqrt{\pi n}} = 1,$$

   using the Stirling approximation $n! \sim \sqrt{2\pi n}\cdot (n/\mathrm{e})^n$.

3. *Prove $\{a_n\}_n$ is an increasing sequence.*

4. *From (b) and (c), conclude that $a_n \le 1$, implying $\binom{2n}{n} \le U_n$.*

5. *Prove $\{b_n\}_n$ is a decreasing sequence.*

6. *From (b) and (e), conclude that $b_n \ge 1$, implying $\binom{2n}{n} \ge L_n$.*

Remark: What did we achieve from this exercise? We started from the asymptotic estimate $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$. From this asymptotic estimate, we obtained explicit upper and lower bounds. We learned a powerful general technique to translate asymptotic estimates into explicit upper and lower bounds automatically.

Solution.

Problem 4: Top Eigenvalue of Random Matrices. (25 pts)

Let M be an $n \times n$ symmetric random matrix such that $\{M_{ij} : i \geq j\}$ are i.i.d. symmetric Bernoulli random variables with $\Pr[M_{ij} = \pm 1] = \frac{1}{2}$. Let $\lambda_{\max}(M)$ denote the largest eigenvalue of M, and let $v_{\max}(M)$ be a corresponding unit eigenvector. Recall the variational characterization

$$\lambda_{\max}(M) = \sup_{v \in B_2} \langle v, Mv \rangle,$$

where $B_2 := \{v \in \mathbb{R}^n : \|v\|_2 \leq 1\}$.

1. Fix indices $i \geq j$. Define $M^{-(ij)}$ to be the symmetric matrix obtained by choosing the entry $M_{ij}^{-(ij)} = M_{ji}^{-(ij)} \in \{-1, 1\}$ so as to minimize $\lambda_{\max}(M^{-(ij)})$, while keeping all other entries fixed (i.e., $M_{kl}^{-(ij)} = M_{kl}$ for $\{k, l\} \neq \{i, j\}$). Define

   $$D_{ij}^{-} \lambda_{\max}(M) := \lambda_{\max}(M) - \lambda_{\max}(M^{-(ij)}).$$

   Show that

   $$D_{ij}^{-} \lambda_{\max}(M) \leq \langle v_{\max}(M), (M - M^{-(ij)}) v_{\max}(M) \rangle.$$

2. Use the fact that M and $M^{-(ij)}$ differ only in the (i, j) and (j, i) entries to prove that

   $$D_{ij}^{-} \lambda_{\max}(M) \leq 4 \, |v_{\max}(M)_i| \, |v_{\max}(M)_j|.$$

3. Conclude that

   $$\sum_{i,j=1}^{n} \left(D_{ij}^{-} \lambda_{\max}(M) \right)^2 \leq 16.$$

4. (Variance) Using the tensorization of variance to prove that

   $$\operatorname{Var}[\lambda_{\max}(M)] \leq 16.$$

5. (Concentration) Apply the bounded difference inequality to prove that for all $t \geq 0$,

   $$\Pr\left[\lambda_{\max}(M) - \mathbb{E}\lambda_{\max}(M) \geq t\right] \leq \exp\left(-\frac{t^2}{64} \right).$$

Solution.

Problem 5: Random Graphs. (15 pts)

Let $G \sim G(n, p)$ be an Erdős–Rényi random graph on vertex set $[n] = \{1, \ldots, n\}$, where each edge appears independently with probability p. A coloring of the graph is the assignment of a color to each vertex such that every pair of vertices connected by an edge have distinct colors. The chromatic number $\chi(G)$ is the minimal number of colors needed to color the graph.

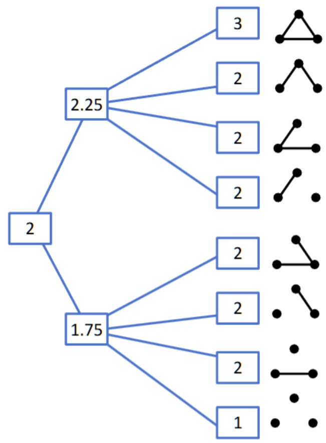

Figure 1: Vertex-exposure martingale for the chromatic number of $G(n, p)$ with $n = 3$. The martingale is obtained by starting at the leftmost point and splitting at each branch with equal probability. (Source: Yufei Zhao)

1. (Vertex exposure martingale) We can reveal the random graph $G(n, p)$ by first fixing an order on all the vertices and, at the i-th step, with $0 \le i \le n$, revealing all edges whose endpoints are contained in the first i vertices. This process produces a martingale $M_0, M_1, \ldots, M_n$ where M_i is the conditional expectation of $\chi(G)$ after revealing whether there are edges connected to the first i vertices.
   Show that $\{M_k\}_{k=0}^n$ is a martingale and that $M_0 = \mathbb{E}[\chi(G)]$ and $M_n = \chi(G)$.

2. Prove that for every $k \in \{1, \ldots, n\}$,

$$\|M_k - M_{k-1}\|_\infty \le 1.$$

   (Hint: compare two graphs that differ only in the edges incident to vertex k and show that their chromatic numbers differ by at most 1.)

3. Apply McDiarmid's inequality to show that for all $t \ge 0$,

$$\Pr\left[|\chi(G) - \mathbb{E}\chi(G)| \ge t\sqrt{n}\right] \le 2e^{-2t^2}.$$

Solution.